

ALEXANDER MODULES OF LINKS WITH ALL LINKING NUMBERS ZERO

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ABSTRACT. In this paper we characterize the Alexander modules of links resulting from a surgical modification on the trivial link of any number of components. Using the presentation matrix obtained, we derive some properties of the Alexander polynomials of such links.

1. In [B, N, R, L, L-1 and L-2] the process of surgical modification of classical links is studied. [B and N] consider the case of two-component links and give answers in terms of a change in the presentation matrix of the Alexander invariant B . The general case of the Alexander module A (of which B is a submodule) is studied in [L-1] but only after A has been localized.

In the present paper, the effect of a surgical modification of the trivial link of any multiplicity is studied. The primary results are Theorems A and B. In Theorem A, a presentation matrix for such links is given. In Theorem B, it is shown that any matrix satisfying the requirements of Theorem A can be realized as the presentation matrix of a link all of whose linking numbers are zero.

Finally, the matrix given in Theorem A is examined in more detail. The Alexander polynomial of such links is determined from the matrix and is shown to be highly divisible by the augmentation ideal.

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2. A classical link of multiplicity n is a smooth, ordered, oriented submanifold of S^3 consisting of n disjoint circles, say $L = L_1 \cup \cdots \cup L_n$. Let N_i be a tubular neighborhood of L_i chosen so that $N_i \cap N_j = \emptyset$ if $i \neq j$. Choose a basepoint $x_i \in \partial N_i$ and meridian and longitude circles in ∂N_i which intersect at x_i and are oriented so that the intersection number $m_i \cdot l_i = +1$ in ∂N_i . In addition, l_i is to be oriented in the same direction as L_i with linking number $\text{lk}(l_i, L_i) = 0$.

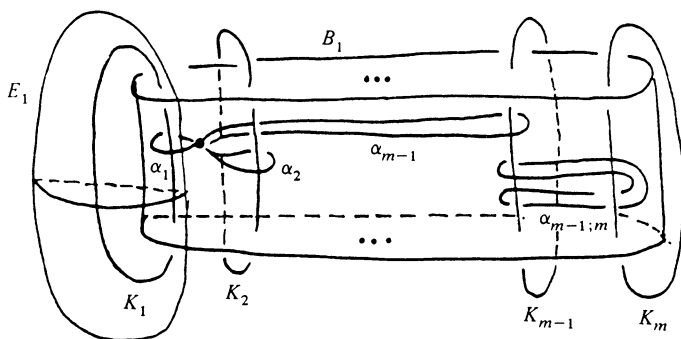
Let x_0 be a basepoint in the interior of $X = \overline{S^3 - \bigcup N_i}$. For each i , choose an oriented arc a_i from x_0 to x_i in X . The elements α_i and β_i in $\pi_1 X$ represented by $a_i * m_i * a_i^{-1}$ and $a_i * l_i * a_i^{-1}$ are referred to as the meridians and longitudes of the link L . The pair (α_i, β_i) is well defined up to simultaneous conjugation by an element of $\pi_1 X$.

By Alexander duality, $H_1(X) \approx \mathbf{Z}^n$ with a basis defined by the meridians of the link. This basis allows the identification of the group ring $\mathbf{Z}[H_1(X)]$ with $\Lambda_n = \mathbf{Z}[t_1, t_1^{-1}, \dots, t_n, t_n^{-1}]$. There is a natural involution of Λ_n (denoted with an

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FIGURE

overbar) which maps $t_i \rightarrow t_i^{-1}$. The augmentation of Λ_n is given by $\varepsilon: \Lambda_n \rightarrow \mathbf{Z}$ where $\varepsilon(t_i) = 1$.

The homomorphism $h: \pi_1 X \rightarrow H_1(X)$ defines a regular covering space, the universal abelian covering $p: \tilde{X} \rightarrow X$ with \mathbf{Z}^n as the group of covering translations. $A = A(L) = H_1(\tilde{X}, \tilde{x}_0)$ is a module over Λ_n referred to as the Alexander module of L . An important submodule of A is $B = H_1(\tilde{X})$. These modules are related by the Crowell exact sequence [C, H]

$$(1) \quad 0 \rightarrow B \xrightarrow{\psi} A \xrightarrow{\phi} M \rightarrow 0$$

where ϕ is the boundary homomorphism $\phi: A \rightarrow H_0(\tilde{X})$. Here, $H_0(\tilde{X})$ is identified with Λ_n by a choice of basepoint $\tilde{x}_0 \in \tilde{x}_0$ and M is the augmentation ideal of Λ_n generated by $t_1 - 1, \dots, t_n - 1$.

We will use the technique of surgical modification as described in [L and L-1]. Let $S = \{\sigma_1, \dots, \sigma_r\}$ be a collection of smooth disjoint circles imbedded in S^3 satisfying:

- (i) $\sigma_i \cap L_j = \emptyset$ for all i, j ,
- (ii) $\text{lk}(\sigma_i, L_j) = 0$ for all i, j ,
- (iii) $\{\sigma_i\}$ considered as a link in S^3 is trivial.

Choose a normal framing which winds around each σ_i once in the positive direction and do surgery on S^3 along $\{\sigma_i\}$. The resulting manifold Σ is diffeomorphic to S^3 by (iii). The image of L in Σ is the surgical modification L_S .

From now on, we let L_0 denote the trivial link of n -components in S^3 with $X_0 = \overline{S^3} - \bigcup N_i$ and universal abelian cover \tilde{X}_0 . Let μ_i denote a lift of the i th meridian of L_0 to \tilde{X}_0 . We note that $\sum_{i=1}^k \lambda_i \mu_i$ is a cycle in \tilde{X}_0 if and only if $\sum_{i=1}^k (t_i - 1) \lambda_i = 0$. This is a direct consequence of the Crowell sequence (1). $A_0 = A(L_0)$ is a free Λ_n -module generated by e_1, \dots, e_n where e_i is represented by μ_i . $B_0 = B(L_0)$ is generated by elements e_{ij} ($i < j$) where $\psi(e_{ij}) = (t_j - 1)e_i + (1 - t_i)e_j$. There are relations among the e_{ij} . Namely, for $i < j < k$, $(t_k - 1)e_{ij} + (1 - t_j)e_{ik} + (t_i - 1)e_{jk} = 0$. We also have $H_2(\tilde{X}_0, \tilde{x}_0) \approx H_2(\tilde{X}_0)$ is generated by $(n - 1)$ two-spheres. Let E_i denote the two-sphere in S^3 which contains only the i th component of L_0 and excludes the basepoint of X_0 . The intersection pairings of E_i and e_j can be determined by considering E_i as an element of $H_2(\tilde{X}_0 - \tilde{x}_0) \approx H_C^1(\tilde{X}_0, \tilde{x}_0)$ and are given by $E_i \cdot e_j = (t_i - 1)\delta_{ij}$. In S^3 , these elements can be represented as shown

in the Figure (see [L-2] for more details). Note that $\{e_{ij}\}_{i < j}$ may be represented so that when considered as a link, it is trivial. For L_S , we adopt the notation $X = \overline{\Sigma} - \bigcup N_i$ with universal abelian cover \tilde{X} .

With the conditions as above we have

THEOREM A. *Let L_S be a surgical modification of L_0 , the trivial link of n -components. The Alexander module of L , $A = A(L)$ has a presentation with generators $e''_1, \dots, e''_n, \alpha''_1, \dots, \alpha''_r$ and relations*

$$\sum_{j=1}^n (t_i - 1) \overline{\lambda_{ji}} \alpha''_j = 0 \quad \text{for } 1 \leq i \leq n$$

and

$$\sum_{j=1}^n \lambda_{ij} e''_j + \sum_{j=1}^k \gamma_{ij} \alpha''_j = 0 \quad \text{for } 1 \leq i \leq r$$

where

- (i) $\sum_{j=1}^n (t_j - 1) \lambda_{ij} = 0$,
- (ii) $\gamma_{ij} = \overline{\gamma_{ji}}$,
- (iii) $\varepsilon(\gamma_{ij}) = \delta_{ij}$.

PROOF. The proof is similar to that of Bailey's Theorem as given in [L-2].

Let $S = \{\sigma_1, \dots, \sigma_r\}$ be the set of circles used to perform the surgical modification and let $\tilde{\sigma}_{i1}$, be any lift of σ_i to \tilde{X}_0 . Let Y be the complement of S in X_0 and \tilde{Y} the covering of Y inherited from \tilde{X}_0 . In order to compute $H_1(\tilde{Y}, \tilde{x}_0)$ consider the exact sequence of the triple $(\tilde{X}_0, \tilde{Y}, \tilde{x}_0)$.

$$(2) \quad \rightarrow H_2(\tilde{X}_0, \tilde{x}_0) \rightarrow H_2(\tilde{X}_0, \tilde{Y}) \rightarrow H_1(\tilde{Y}, \tilde{x}_0) \rightarrow A_0 \rightarrow 0.$$

Since A_0 is free and $H_2(\tilde{X}_0, \tilde{Y})$ is free on $\{\alpha_i\}$,

$$H_1(\tilde{Y}, \tilde{x}_0) \approx A_0 \oplus H_2(\tilde{X}_0, \tilde{Y}) / \text{Im } H_2(\tilde{X}_0, \tilde{x}_0).$$

Since σ_i lifts to an imbedded circle $\tilde{\sigma}_{i1}$ in \tilde{X}_0 , (1) implies

$$(3) \quad \tilde{\sigma}_{i1} = \sum_{j=1}^n \lambda_{ij} e_j \quad \text{with} \quad \sum_{j=1}^n (t_j - 1) \lambda_{ij} = 0.$$

From this, it is clear that $H_1(\tilde{Y}, \tilde{x}_0)$ is generated by e'_1, \dots, e'_n where $e'_i \rightarrow e_i$ under the inclusion $Y \rightarrow X_0$ and by $\alpha_1, \dots, \alpha_r$ where α_i is represented by a small circle which links $\tilde{\sigma}_{i1}$ simply.

The relations in $H_1(\tilde{Y}, \tilde{x}_0)$ are given by the images of elements in $H_2(\tilde{X}_0, \tilde{x}_0)$. Since $H_2(\tilde{X}_0, \tilde{x}_0)$ is freely generated by $(n-1)$ of the E_i , there is one relation for each E_i given by the intersection pairing. By using all of the E_i , we obtain n relations, R_1, \dots, R_n , one of which is redundant. That is,

$$(4) \quad R_i: \sum_{j=1}^r \gamma_{ij} \alpha_j = 0 \quad \text{where } \gamma_{ij} = E_i \cdot \tilde{\sigma}_{j1}.$$

Using (3) we obtain

$$\gamma_{ij} = E_i \cdot \left(\sum_{k=1}^n \lambda_{jk} e_k \right) = \sum_{k=1}^n \overline{\lambda_{jk}} (E_i \cdot e_k) = (t_i - 1) \overline{\lambda_{ji}}.$$

Thus, the relations are

$$(5) \quad \sum_{j=1}^r (t_i - 1) \overline{\gamma_{ji}} \alpha_j = 0, \quad 1 \leq i \leq n,$$

and $H_1(\tilde{Y}, \tilde{x}_0)$ has a presentation with generators $e'_1, \dots, e'_n, \alpha_1, \dots, \alpha_k$ and relations (5). (Note that the rows of the corresponding presentation matrix are linearly dependent since $\sum_{i=1}^n E_i = 0$ in $H_2(\tilde{X}_0, \tilde{x}_0)$.)

To compute A , we examine the exact sequence of the triple $(\tilde{X}, \tilde{Y}, \tilde{x}_0)$

$$(6) \quad \rightarrow H_2(\tilde{X}, \tilde{Y}) \xrightarrow{\partial^*} H_1(\tilde{Y}, \tilde{x}_0) \xrightarrow{i_*} A \rightarrow 0.$$

From this, we see that $A \approx H_1(\tilde{Y}, \tilde{x}_0)/\partial_* H_2(\tilde{X}, \tilde{Y})$ is generated by $e''_1, \dots, e''_n, \alpha'_1, \dots, \alpha'_r$ the images of $e'_1, \dots, e'_n, \alpha_1, \dots, \alpha_r$ under the inclusion $Y \rightarrow X$. In addition, there are new relations obtained by attaching a disc along $\tilde{\sigma}'_{i1}$, where the circle $\tilde{\sigma}'_{i1}$ is the result of translating $\tilde{\sigma}_{i1}$ along the vector field used in the surgery. $H_2(\tilde{X}, \tilde{Y})$ is generated by d_1, \dots, d_r where d_i can be represented by a disc whose boundary is $\tilde{\sigma}'_{i1}$. The additional relations are given by

$$(7) \quad R_i: i_* \partial_*(d_i) = 0.$$

That is,

$$(8) \quad \sum_{j=1}^n \lambda_{ij} e''_j + \sum_{j=1}^r \gamma'_{ij} \alpha'_j = 0 \quad \text{for some } \gamma'_{ij}.$$

That the coefficient of e''_j is λ_{ij} follows from (3). From (1) and (3) we may write

$$\tilde{\sigma}'_i = \sum_{s < t} \gamma_{ist} e_{st} + \sum_{j=1}^r \gamma'_{ij} \alpha''_j.$$

We need to show that the correct original choice of $\{e_{st}\}_{s < t}$ and corresponding $\{e_j\}$ results in the following properties:

$$(9) \quad \text{(i) } \gamma'_{ij} = \overline{\gamma'_{ji}}, \quad \text{(ii) } \varepsilon(\gamma'_{ij}) = \delta_{ij}.$$

We need to use the following lemma from [L-2].

LEMMA. Suppose M is a compact oriented 3-manifold, $\tilde{M} \rightarrow M$ a regular covering with π as the group of covering transformations. Let T_1, \dots, T_n be tori components of ∂M which lift to $\tilde{T}_i \subseteq \tilde{M}$ trivially covering T_i , for each i . Let α_i, β_i be the canonical generators of $H_1(\tilde{T}_i)$ represented by meridian and longitude circles satisfying $\alpha_i \cdot \alpha_j = 0 = \beta_i \cdot \beta_j$ and $\alpha_i \cdot \beta_j = \delta_{ij}$. If

$$\sum_j \lambda_{ij} i_*(\alpha_j) + \sum_j \mu_{ij} i_*(\beta_j) = 0, \quad i = 1, \dots, n,$$

is any set of relations in $H_1(\tilde{M})$, $i: \tilde{T}_i \subseteq \tilde{M}$, then, for any i, j

$$\sum_s \lambda_{is} \overline{\mu_{js}} = \sum_s \mu_{is} \overline{\lambda_{js}} \quad \text{where } \mu \rightarrow \bar{\mu}$$

is the usual conjugation in $\mathbb{Z}\pi$.

To apply the lemma, we remove tubular neighborhoods of the loops τ_{ij} , representing e_{ij} , from Y to obtain a new manifold W . The new equations are

$$(10) \quad \tilde{\sigma}'_{i0} = \sum_{s < t} \gamma_{ist} e_{st0} + \sum_J \gamma'_{ij} \alpha_{j0} + \sum_{s < t} \overline{\mu_{ist}} C_{st} \quad \text{in } H_1(\widetilde{W})$$

where

- (i) C_{st} is represented by a meridian of the newly removed neighborhood of τ_{st} ,
- (ii) e_{sto} is represented by $\tilde{\tau}'_{st}$, a translate of $\tilde{\tau}_{st}$ into \widetilde{W} chosen so that $\text{lk}(\tau'_{st}, \tau_{st}) = 0$ in S^3 ,
- (iii) $\alpha_{j0} \rightarrow \alpha''_j$ and $\tilde{\sigma}'_{i0} \rightarrow \tilde{\sigma}'_i$.

Applying the lemma to (10) gives

$$(11) \quad \gamma'_{ij} + \sum_{s < t} \gamma_{ist} \mu_{jst} = \overline{\gamma'_{ji}} + \sum_{s < t} \overline{\mu_{ist}} \overline{\gamma_{jst}}$$

assuming that $\{\alpha''_j\}$ and $\{c_{st}\}$ are oriented correctly. Now we replace the original choice of e_{st} with $\varepsilon_{st} = e_{st} - \sum_{j=1}^r \mu_{jst} \alpha''_j$. We note that

$$(12) \quad \begin{aligned} \tilde{\sigma}'_i &= \sum_{s < t} \gamma_{ist} \left(\varepsilon_{st} + \sum_{j=1}^r \mu_{jst} \alpha''_j \right) + \sum_{j=1}^r \gamma'_{ij} \alpha''_j \\ &= \sum_{s < t} \gamma_{ist} \varepsilon_{st} + \sum_{j=1}^r \left(\gamma'_{ij} + \sum_{s < t} \gamma_{ist} \mu_{jst} \right) \alpha''_j. \end{aligned}$$

Hence γ'_{ij} is replaced by $\gamma_{ij} = \gamma'_{ij} + \sum_{s < t} \gamma_{ist} \mu_{jst}$ so property (9)(i) is satisfied.

To verify (9)(ii) we project everything to $W \subseteq S^3$ and note

(a)

$$\begin{aligned} \varepsilon(\mu_{ist}) &= \text{lk} \left(\tau_{st}, \sigma_i - \sum_{u < v} \varepsilon(\gamma_{iuv}) \tau'_{uv} \right) \\ &= \text{lk}(\tau_{st}, \sigma_i) - \sum_{u < v} \varepsilon(\gamma_{iuv}) \text{lk}(\tau_{st}, \tau'_{uv}) \\ &= \text{lk}(\tau_{st}, \sigma_i) \quad (\text{recalling that the link formed by } \{\tau_{st}\} \text{ is trivial}), \end{aligned}$$

(b)

$$\begin{aligned} \varepsilon(\gamma'_{ij}) &= \text{lk} \left(\sigma_j, \sigma'_i - \sum_{u < v} \varepsilon(\gamma_{iuv}) \tau'_{uv} \right) \\ &= \text{lk}(\sigma_j, \sigma'_i) - \sum_{u < v} \varepsilon(\gamma_{iuv}) \text{lk}(\sigma_j, \tau'_{uv}) \\ &= \delta_{ij} - \sum_{u < v} \varepsilon(\gamma_{iuv}) \varepsilon(\mu_{juv}). \end{aligned}$$

Together (a) and (b) imply

$$\varepsilon(\gamma_{ij}) = \varepsilon(\gamma'_{ij}) - \sum_{u < v} \gamma_{iuv} \mu_{juv} = \delta_{ij}$$

and property (9)(ii) is satisfied concluding the proof of Theorem A.

We now prove the following theorem.

THEOREM B. Let (λ_{ij}) be an $(m \times m)$ -matrix over Λ_n satisfying

- (i) $\lambda_{ij} = 0$ for $1 \leq i, j \leq n$,
- (ii) $\sum_{j=1}^r (t_i - 1)\lambda_{ij} = 0$ for $i > n$,
- (iii) $\lambda_{ij} = (t_i - 1)\overline{\lambda_{ji}}$ for $i > n$ and $1 \leq j \leq n$,
- (iv) $\lambda_{ij} = \overline{\lambda_{ji}}$ for $i, j > n$,
- (v) $\varepsilon(\lambda_{ij}) = \delta_{ij}$ for $i, j > n$.

Then there exists a link with all linking numbers zero, whose module A has a presentation $\{\alpha_1, \dots, \alpha_m: \sum_{j=1}^m \lambda_{ij}\alpha_j = 0\}$.

PROOF. The proof is similar to that of Theorem A, and we use the same notation. Let X_0 be the complement of the n -component trivial link with generators e_i of $H_1(\tilde{X}_0, \tilde{x}_0)$. Let $\{\sigma_i\}_{i=n+1}^m$ be disjoint imbedded circles in X_0 which lift to $\{\tilde{\sigma}_i\}$ in \tilde{X}_0 such that $\tilde{\sigma}_i$ represents $\sum_{j=1}^n \lambda_{ij}e_j$. Properties (ii) and (iii) insure that σ_i is an imbedded circle in \tilde{X}_0 . Furthermore, situate $\{\sigma_i\}$ so that $\{\sigma_i\}$ is a trivial link in S^3 . Give each σ_i the normal framing which winds around once and do surgery. $H_1(\tilde{Y}, \tilde{x}_0)$ and $H_1(\tilde{X}, \tilde{x}_0)$ are computed precisely as before. We obtain relations

$$(13) \quad \sum_{j=1}^n \lambda_{ij}e_j'' + \sum_{j=n+1}^m \lambda'_{ij}\alpha_j'' = 0$$

for some $\gamma'_{ij} \in \Lambda_n$. The γ'_{ij} may not be the λ_{ij} specified in the hypothesis. However, by the proper choice of e_{ij} and e_i we have $\gamma'_{ij} = \overline{\gamma'_{ji}}$ and $\varepsilon(\gamma'_{ij}) = \delta_{ij}$. Finally, the γ'_{ij} can be altered in order to realize the specified λ_{ij} by using a combination of the elementary changes described in [L-2].

3. The Alexander polynomial Δ of the link can be determined from a presentation matrix for A . It is the greatest common divisor of all the $(n + m - 1)$ -minors. [T] has shown that $\Delta \in M^{2n-3}$ in general when all linking numbers are zero. In Theorem H, we show that in our situation $\Delta \in M^{3n-4}$. Thus, the Borromean rings are not a surgical modification of the trivial 3-component link since the Alexander polynomial of the Borromean rings is $(t_1 - 1)(t_2 - 1)(t_3 - 1)$. We adopt the following notation.

$$M = \left[\begin{array}{ccc|ccc} & & & (t_1 - 1)\overline{\lambda_{11}} & \dots & (t_1 - 1)\overline{\lambda_{m1}} \\ & & & \vdots & & \vdots \\ (0) & & & (t_n - 1)\overline{\lambda_{1n}} & \dots & (t_n - 1)\overline{\lambda_{mn}} \\ \hline \lambda_{11} & \dots & \lambda_{1n} & & & \\ \vdots & & \vdots & & & \\ \lambda_{m1} & \dots & \lambda_{mn} & & & \end{array} \right] \quad (\gamma_{ij})$$

where $\sum_j (t_j - 1)\lambda_{ij} = 0$. Let M_{ij} denote the $(n + m - 1)$ -submatrix obtained by deleting the i th row and j th column of M . Then,

- PROPOSITION C.** (i) $|M_{ij}| = 0$ if $i > n$ or $j > n$,
 (ii) $(t_j - 1)|M_{ii}| = \pm(t_i - 1)|M_{ij}|$,
 (iii) $|M_{ij}| = (\text{unit})|M_{jj}|$ for $i, j \leq n$, $i \neq j$.

Thus, M_{ii} and M_{jj} are related.

PROOF. (i) If $i > n$ (resp. $j > n$), then M_{ij} contains the first n rows (resp. columns) of M which are linearly dependent since $\sum_j (t_j - 1)\lambda_{ij} = 0$.

(ii) If $i, j \leq n$, column 1 can be replaced by a linear combination of the first n columns of M excluding column 1. More precisely, denote by $\hat{}$ that a row (column) has been deleted. Since $\sum_j (t_j - 1)\lambda_{ij} = 0$ for each i , we may write (using the relation $\sum_r (t_r - 1)\lambda_{kr} = 0$ in the last step)

$$\begin{aligned}
 |M_{ii}| &= \prod_{k \neq i} (t_k - 1) \left| \begin{array}{ccc|ccc} & & & \overline{\lambda_{11}} & \dots & \overline{\lambda_{m1}} \\ & & & \vdots & & \vdots \\ & (0) & & \overline{\lambda_{1n}} & \dots & \overline{\lambda_{mn}} \\ \hline \overline{\lambda_{11}} & \dots & \overline{\lambda_{1n}} & & & \\ \vdots & & \vdots & & & \\ \lambda_{m1} & \dots & \lambda_{mn} & & & \end{array} \right|^i \\
 &= \left| \begin{array}{ccc|ccc} & & & \overline{\lambda_{11}} & \dots & \overline{\lambda_{m1}} \\ & & & \vdots & & \vdots \\ & (0) & & \overline{\lambda_{1n}} & \dots & \overline{\lambda_{mn}} \\ \hline (t_1 - 1)\overline{\lambda_{11}} & \dots & (t_n - 1)\overline{\lambda_{1n}} & & & \\ \vdots & & \vdots & & & \\ (t_1 - 1)\lambda_{m1} & \dots & (t_n - 1)\lambda_{mn} & & & \end{array} \right|^i \\
 &= \left| \begin{array}{ccc|ccc} & & & \overline{\lambda_{11}} & \vdots & \overline{\lambda_{1n}} \\ & & & \vdots & & \vdots \\ & (0) & & \overline{\lambda_{1n}} & \dots & \overline{\lambda_{mn}} \\ \hline (t_1 - 1)\overline{\lambda_{11}} & \dots & \sum_{k \neq i} (1 - t_k)\lambda_{1k} & \dots & (t_n - 1)\overline{\lambda_{1n}} & \\ \vdots & & \vdots & & \vdots & \\ (t_1 - 1)\lambda_{m1} & \dots & \sum_{k \neq i} (1 - t_k)\lambda_{mk} & \dots & (t_n - 1)\lambda_{mn} & \end{array} \right|^i
 \end{aligned}$$

Now add to column j , the sum of the other $(n - 2)$ initial columns.

$$\begin{aligned}
 |M_{ii}| &= \left| \begin{array}{ccc|ccc} & & & \overline{\lambda_{11}} & \vdots & \overline{\lambda_{1n}} \\ & & & \vdots & & \vdots \\ & (0) & & \overline{\lambda_{1n}} & \dots & \overline{\lambda_{mn}} \\ \hline (t_1 - 1)\overline{\lambda_{11}} & \dots & (1 - t_i)\overline{\lambda_{1i}} & \dots & (t_n - 1)\overline{\lambda_{1n}} & \\ \vdots & & \vdots & & \vdots & \\ (t_1 - 1)\lambda_{m1} & \dots & (1 - t_i)\lambda_{mi} & \dots & (t_n - 1)\lambda_{mn} & \end{array} \right|^i \\
 &= - \prod_{k \neq j} (t_k - 1) \left| \begin{array}{ccc|ccc} & & & \overline{\lambda_{11}} & \dots & \overline{\lambda_{m1}} \\ & & & \vdots & & \vdots \\ & (0) & & \overline{\lambda_{1n}} & \dots & \overline{\lambda_{mn}} \\ \hline \overline{\lambda_{11}} & \dots & \overline{\lambda_{1n}} & & & \\ \vdots & & \vdots & & & \\ \lambda_{m1} & \dots & \lambda_{mn} & & & \end{array} \right|^i.
 \end{aligned}$$

Thus,

$$(t_j - 1)|M_{ii}| = -(t_i - 1) \left| \begin{array}{ccc|ccc} & & & (t_1 - 1)\overline{\lambda_{11}} & \dots & (t_1 - 1)\overline{\lambda_{m1}} \\ & (0) & & \vdots & & \vdots \\ \overline{\lambda_{11}} & \dots & \overline{\lambda_{1n}} & (t_n - 1)\overline{\lambda_{1n}} & \dots & (t_n - 1)\overline{\lambda_{mn}} \\ \vdots & & \vdots & & & \\ \lambda_{m1} & \dots & \lambda_{mn} & & & \end{array} \right|^i$$

$\underbrace{\hspace{10em}}_{\substack{j \\ \gamma_{ij}}}$

That is, $(t_j - 1)|M_{ii}| = \pm(t_i - 1)|M_{ij}|$, the sign depending on the parity of the permutation of the first $(n - 1)$ columns.

(iii) This follows from a calculation similar to that of (ii) using the first $(n - 1)$ rows of M_{ij} and the conjugate relation $\sum_j (t_j^{-1} - 1)\overline{\lambda_{ij}} = 0$. The details are left to the reader.

COROLLARY D. *If $m < (n - 2)$, $|M_{ii}| = 0$ for all i . Consequently, the Alexander polynomial of the corresponding link is zero.*

PROOF. By Theorem C, it is only necessary to consider M_{ii} for $i \leq n$. This matrix has the form

$$\begin{pmatrix} 0_{(n-1) \times (n-1)} & *_{(n-1) \times m} \\ *_{m \times (n-1)} & *_{m \times m} \end{pmatrix}.$$

The determinant is the sum of terms obtained by choosing an entry from each column and row. From the first $(n - 1)$ columns, we must choose $(n - 1)$ entries, each from a different row. However, there are at most $(n - 2)$ nonzero choices, so at least one zero must be chosen. Thus, each term is zero, hence so is the sum.

An easy calculation gives the following corollaries which are also proven more generally in [C-1].

COROLLARY E. *If $i < n$, then $|M_{ii}| = (t_i - 1)[\prod_k (t_k - 1)]\Delta$ (up to a unit) for some $\delta \in \Lambda_n$.*

COROLLARY F. *The Alexander polynomial of the link L_S is $A(t_1, \dots, t_n) = [\prod_k (t_k - 1)]\Delta$.*

We need

LEMMA G. *If $\sum_j (t_j - 1)\lambda_j = 0$, then λ_j is in the ideal generated by the $(t_k - 1)$ for $k \neq j$. That is, $\lambda_j \in ((t_1 - 1), \dots, (\widehat{t_j - 1}), \dots, (t_n - 1))$.*

PROOF. From the Koszul complex

$$\begin{aligned} &\rightarrow \dots \binom{n}{2} \Lambda_n \rightarrow n\Lambda_n \rightarrow \Lambda_n \rightarrow \mathbf{Z} \rightarrow 0, \\ &\sum_j (t_j - 1)\lambda_j = 0 \quad \text{implies} \quad \lambda_j = \sum_i a_{ij}(t_i - 1) \end{aligned}$$

with $a_{ij} = -a_{ji}$. Hence $a_{ii} = 0$ and the result follows.

Using Lemma G we prove the following theorem.

THEOREM H. *The Alexander polynomial A of any link obtained by a surgical modification of the trivial link is in M^{3n-4} where M is the augmentation ideal.*

PROOF. It is enough to consider

$$|M_{11}| = \left[\prod_{k=2}^n (t_k - 1) \right] \Delta_1 = (t_1 - 1) \left[\prod_{k=1}^n (t_k - 1) \right] \Delta$$

where

$$\Delta_1 = \begin{vmatrix} 0 & \overline{(\lambda_{ji})} \\ (\lambda_{ij}) & * \end{vmatrix} \quad (j \neq 1).$$

Each nonzero summand of Δ_1 has the form

$$(\lambda_{i_2 2} \dots \lambda_{i_m m}) (\overline{\lambda_{2 k_2}} \dots \overline{\lambda_{m k_m}}) \quad (\text{product of entries in } *)$$

where $i_r \neq i_s$ and $k_r \neq k_s$. Since each $\lambda_{ij} \in M$, the summands are in $M^{n-1} \cdot M^{n-1} = M^{2n-2}$. Since $\Delta_1 = (t_1 - 1)^2 \Delta$, $\Delta \in M^{2n-4}$. Thus

$$A = \left[\prod (t_k - 1) \right] \Delta \in M^n \cdot M^{2n-4} = M^{3n-4}.$$

We conclude with a result related to one of Kidwell [K]. Suppose $n = 3$ and A is the Alexander polynomial of any 3-component link for which all linking numbers are zero. Then A can be factored as $A = (t_1 - 1)(t_2 - 1)(t_3 - 1)\Delta$. In [K] it is shown that $|\varepsilon(\Delta)|$ is a perfect square and that any square can occur.

For any surgical modification of the trivial link, we have

THEOREM I. $|\varepsilon(\Delta)| = 0$ ($n \geq 3$).

PROOF. $A \in M^{3n-4}$ and $\Delta \in M^{2n-4}$ and $2n - 4 > 0$.

REFERENCES

- [B] J. H. Bailey, *Alexander invariants of links*, Ph. D. thesis, Univ. of British Columbia, 1977.
- [C] R. H. Crowell, *Corresponding group and module sequences*, Nagoya Math. J. **19** (1961), 27–40.
- [C-1] —, *Torsion in link modules*, J. Math. Mech. **14** (1965), 289–298.
- [H] J. A. Hillman, *Alexander ideals of links*, Lecture Notes in Math., vol. 895, Springer-Verlag, New York, 1981.
- [K] M. Kidwell, *On the Alexander polynomial of certain three component links*, Proc. Amer. Math. Soc. **71** (1978), 351–353.
- [L] J. P. Levine, *A method for generating link polynomials*, Amer. J. Math. **89** (1967), 69–84.
- [L-1] —, *The presentation matrix of a link module*, preprint.
- [L-2] —, *The module of a two-component link*, Comment. Math. Helv. **57** (1982), 377–399.
- [N] Y. Nakanishi, *A surgical view of Alexander invariants of links*, Math. Sem. Notes Kobe Univ. **8** (1980), 199–218.
- [R] D. Rolfsen, *Knots and links*, Publish or Perish, Berkeley, Calif., 1976.
- [T] L. Traldi, *Linking numbers and elementary ideals of links*, Trans. Amer. Math. Soc. **275** (1983).

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